Lecture 9

$\mathfrak{sl}_n\mathbb{C} = \{X \in \mathfrak{gl}_n\mathbb{C} \mid \operatorname{tr}(X) = 0\}$

$\mathfrak{sp}(2n,\mathbb{C}) = \{X \in \mathfrak{gl}_{2n}\mathbb{C} \mid X^TJ + JX = 0\}$ $\quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$

$= \text{Lie(preserves alternating nondegen } \mathbb{C}\text{-bilin form)}$

$\mathfrak{so}(n,\mathbb{C}) = \{X \in \mathfrak{gl}_n\mathbb{C} \mid X^T + X = 0\}$

$= \text{Lie(preserves symmetric nondegen } \mathbb{C}\text{-bilin form)}$

Weyl group $W(V, \langle\rangle, \Phi) = W(\mathfrak{g}, \mathfrak{h})$ = the group generated by the reflections

$$s_\alpha(\varphi) = \varphi - 2\frac{\langle\varphi,\alpha\rangle}{\langle\alpha,\alpha\rangle}\alpha$$

acting on V or $\mathfrak{h}^*$ (or $\mathfrak{h}$, using $\mathfrak{h} \cong \mathfrak{h}^*$ via B)

Facts. Finite (injects into Sym_Φ)
Acts transitively on possible Φ^+ choices.

Presentation. Let Δ = simple roots (rel to Φ^+)
Then W is generated by $\{s_\alpha \mid \alpha \in \Delta\}$.
In fact:

Thm. Let $n(\alpha,\beta)$ = # edges from α to β in the Dynkin diagram. Let

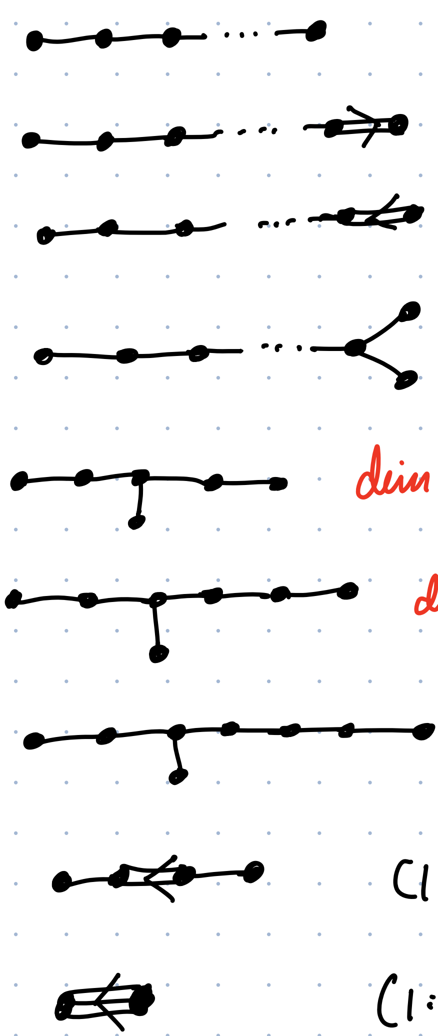
$$m(\alpha,\beta) = \begin{cases} 1 & \alpha = \beta \\ 2 & n(\alpha,\beta) = 0 \\ 3 & n(\alpha,\beta) = 1 \\ 4 & n(\alpha,\beta) = 2 \\ 6 & n(\alpha,\beta) = 3 \end{cases}$$

Then $\langle s_\alpha, \alpha \in \Delta \mid (s_\alpha s_\beta)^{m(\alpha,\beta)}\ \alpha,\beta \in \Delta \rangle \cong W$.

Back to the general classification

$\mathbb{C}$-simple $\mathfrak{g}$ $\Updownarrow$

The connected Dynkin diagrams of reduced root systems:



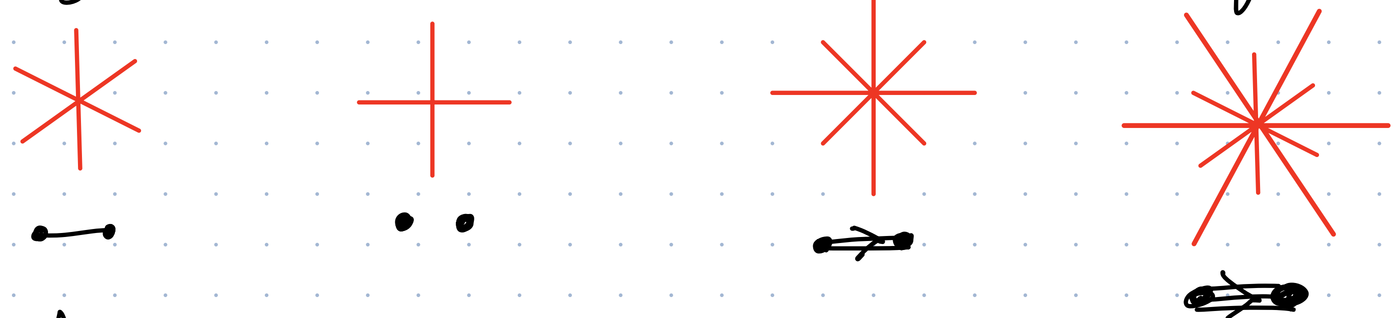
Cartan type			
A_n		$\mathfrak{sl}_{n+1}\mathbb{C}$	$W = \mathrm{Sym}_{n+1}$
B_n		$\mathfrak{so}(2n+1,\mathbb{C})$	$W = \mathrm{Sym}_n \ltimes (\mathbb{Z}/2)^n$
C_n		$\mathfrak{sp}(2n,\mathbb{C})$	$W = \mathrm{Sym}_n \ltimes (\mathbb{Z}/2)^n$
D_n		$\mathfrak{so}(2n,\mathbb{C})$	$W = \mathrm{Sym}_n \ltimes \underbrace{(\mathbb{Z}/2)^{n-1}}_{\text{vectors sum to zero in } (\mathbb{Z}/2)^n.}$
E_6		$\dim\mathfrak{g} = 78$	
E_7		$\dim\mathfrak{g} = 133$	
E_8		$\dim\mathfrak{g} = 248$	
F_4	$(1:2)$	$\dim\mathfrak{g} = 52$	
G_2	$(1:3)$	$\dim\mathfrak{g} = 14$ $\quad \mathfrak{g} \subset \mathfrak{so}(8,\mathbb{C})$	$W = D_6$

E_6, E_7, E_8, F_4: W usually just called "the Weyl group of type ..."

"cartan type"

Cor. Up to isomorphism, the semisimple Lie alg of rank 2 are:

$\mathfrak{sl}_3\mathbb{C}$	$\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}$	$\mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(4,\mathbb{C})$	$\mathfrak{g}_2$



A_2	$A_1 \oplus A_1$	$B_2 \cong C_2$	G_2

Parabolic subalgebras Suppose $\mathfrak{g}, \mathfrak{h}, \Phi^+$ fixed.

$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \subset \mathfrak{g}$ subalgebra. **Solvable** as the coef of Δ get bigger each time you bracket.

Thm. $\mathfrak{b}$ is maximal among solvable subalg.
Any max solv. subalg is conjugate to $\mathfrak{b}$.
means inner aut related.

Def. A maximal solvable subalgebra is called a **Borel** subalgebra.

Cor. Every Cartan is contained in a Borel.

Let $S \subset \Delta$. Let $\Phi_S = \Phi \cap \mathbb{Z}_S^+$.

$$\mathfrak{p}_S := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi_S} \mathfrak{g}_{-\alpha} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma_S} \mathfrak{g}_\alpha$$

Γ_S = roots such that coef rel to Δ are either:
- All positive, or
- All negative and only elements of S are used.

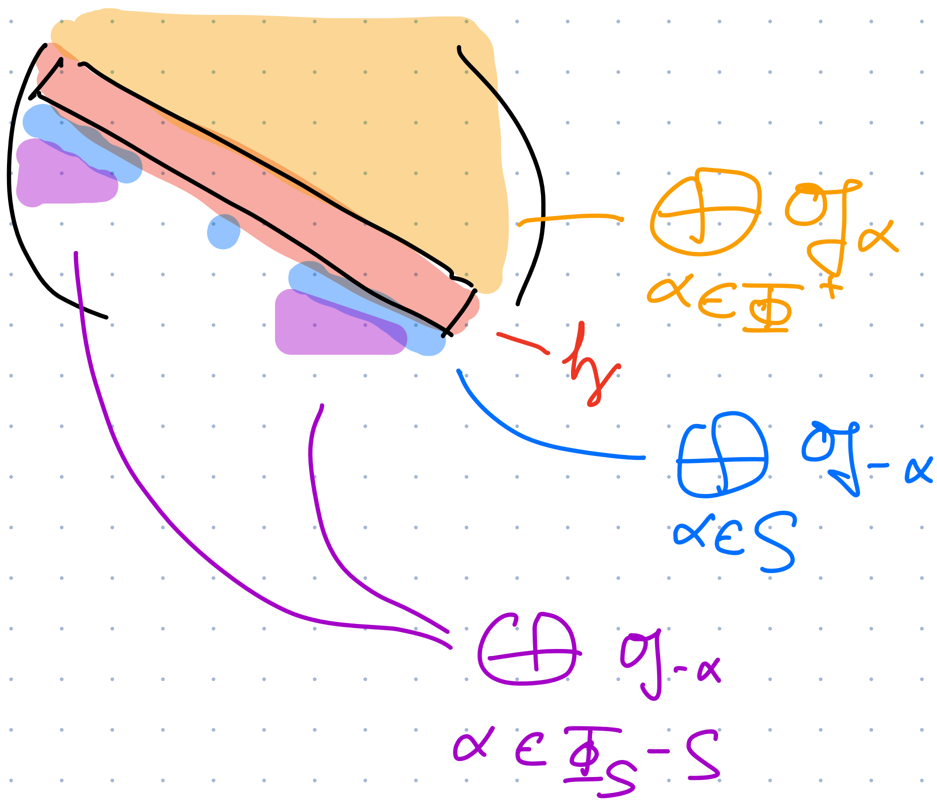
Thm. Any subalg of $\mathfrak{g}$ containing a Borel is conj to one of the form $\mathfrak{p}_S$ for $S \subset \Delta$.

Def. A proper subalg of $\mathfrak{g}$ is called parabolic if it contains a Borel. So: Borel = min parabolic

conj classes max parab $= |\Delta| =$ rank.

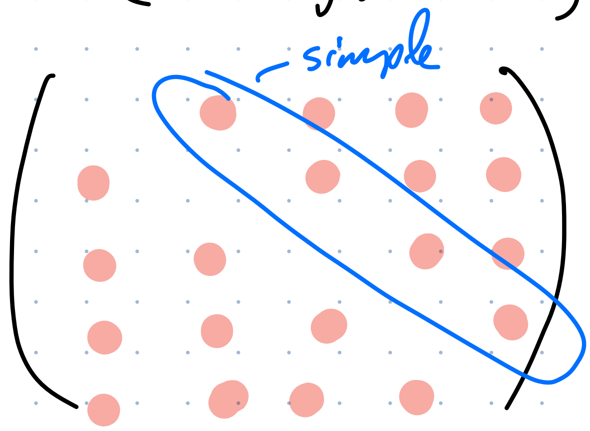
Def. $\{\mathfrak{p}_S\}$ are the standard parabolics (rel $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}$)

Claim. For $\mathfrak{sl}_n\mathbb{C}$ (A_{n-1}), if we use $\{e_i - e_{i+1} \mid i = 1, \dots, n-1\}$ as the simple roots, then $\{\mathfrak{p}_S\} = \{\text{Lie alg of } P_{\underline{d}}\}$.



$\operatorname{span}(e_1 - e_2, e_2 - e_3, e_3 - e_4) \cap \Phi$?

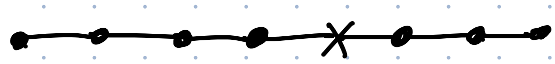
$\{e_i - e_j \mid 1 < i < j < 4\}$

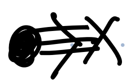


Roots (spaces) for $\mathfrak{sl}_5\mathbb{C} = A_4$.

Diagram $\mathfrak{p} < \mathfrak{g}$ iso type is shown as a Dynkin diagram where every simple root not in S is crossed out.

e.g. $\mathfrak{b} < \mathfrak{so}(9, \mathbb{C})$ 

e.g. a maximal parab in $\mathfrak{sl}_{10}\mathbb{C}$ 

e.g. all parabolics in $\mathfrak{g}_2$: , 

Thm: parabolic subalg of semisimple $\mathfrak{g}$ are self-normalizing.

i.e. $n_{\mathfrak{g}}(\mathfrak{p}) = \{x \in \mathfrak{g} \mid \operatorname{ad}_x(\mathfrak{p}) \subset \mathfrak{p}\} = \mathfrak{p}$.

Normalizers

$\forall g \in G$, $a \mapsto gag^{-1}$ is a smooth map of G, which takes e to e.

Hence get deriv $Ad_g : \underset{\mathfrak{g}}{\overset{\shortparallel}{T_eG}} \to \underset{\mathfrak{g}}{\overset{\shortparallel}{T_eG}}$.

$Ad : G \to Aut(\mathfrak{g})$ homomorphism (deriv is ad)

For a subset $V \subset \mathfrak{g}$, define

$$N_G(V) = \{ g \in G \mid Ad_g(V) \subset V \}$$

Thm. If $E \subset \mathfrak{g}$ is a subspace, then $N_G(E)$ is an embedded Lie subgrp of G with Lie alg $n_{\mathfrak{g}}(E)$.

Pf. The stabilizer of a point in a smooth Lie grp action is a closed Lie subgrp. (It's a preimage of a pt by a const rank map)

Now Ad_G acts on $Gr(\mathfrak{g}, d)$ where $d = \dim(E)$.

Stab of E is the normalizer.

To get Lie alg:

$X \in Lie(N_G(E)) \Rightarrow X \in n_{\mathfrak{g}}(E)$ clear.

$X \in n_{\mathfrak{g}}(E)$: $\exp(tX) \in N_G(E)$ by BCH - $\square$

From Lie alg to Lie groups

We call a complex Lie group G <u>semisimple</u> if $\mathfrak{g} = \mathrm{Lie}(G)$ is a semisimple Lie alg and G has finitely many connected components.

(Such a group is necessarily a linear algebraic group over $\mathbb{C}$.)

A complex Lie subgroup of G is <u>Borel</u> if it is connected, solvable, and maximal for those prop.

Thm. Borel $\Leftrightarrow$ $N_G(\mathfrak{b})$ for $\mathfrak{b} \subset \mathfrak{g}$ a Borel subalg

$\Leftrightarrow$ $\mathrm{Lie}(B) = \mathfrak{b}$ for $\mathfrak{b} \subset \mathfrak{g}$ a Borel.

A complex Lie subgrp is <u>parabolic</u> if it contains a Borel subgroup ($\Rightarrow$ connected)

$\Leftrightarrow$ $\mathrm{Lie}(P) = \mathfrak{p}$ is a parab subalg.

$\Leftrightarrow$ $P = N_G(\mathfrak{p})$ for $\mathfrak{p} \subset \mathfrak{g}$ parab.

Why are normalizers helpful? They give embedded Lie subgrps.